

# Spectral methods in quantum field theory and quantum cosmology

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## Abstract

We review the application of the spectral zeta-function to the 1-loop properties of quantum field theories on manifolds with boundary, with emphasis on Euclidean quantum gravity and quantum cosmology. As was shown in the literature some time ago, the only boundary conditions that are completely invariant under infinitesimal diffeomorphisms on metric perturbations suffer from a drawback, i.e. lack of strong ellipticity of the resulting boundary-value problem. Nevertheless, at least on the Euclidean 4-ball background, it remains possible to evaluate the  $\zeta(0)$  value, which describes in this case a universe which, in the limit of small 3-geometry, has vanishing probability of approaching the cosmological singularity. An assessment of this result is here performed, discussing its physical and mathematical implications.

# 1 Introduction

In the Euclidean functional-integral approach to quantum gravity, one deals with amplitudes written formally as functional integrals over all Riemannian 4-geometries matching the boundary data on (compact) Riemannian 3-geometries  $(\Sigma_1, h_1)$  and  $(\Sigma_2, h_2)$  [1]. To take into account the gauge freedom of the theory, the functional-integral measure also includes suitable ghost fields, described geometrically by a 1-form, hereafter denoted by  $\varphi = \varphi_\mu dx^\mu$ , subject to boundary conditions at  $(\Sigma_1, h_1)$  and  $(\Sigma_2, h_2)$ . Although a rigorous definition of the Feynman sum over all Riemannian 4-geometries with their topologies does not yet exist, the choice of boundary conditions still plays a key role to obtain an elliptic boundary-value problem, which may be applied to the semiclassical analysis of the quantum theory.

In quantum cosmology, it was proposed in [2] and [3] that no boundary conditions should be imposed at the 3-geometry  $(\Sigma_1, h_1)$ , since this might shrink to a point in the case of the quantum state of the universe. One would then have to impose suitable boundary conditions only at  $(\Sigma_2, h_2)$ , by describing the quantum state of the universe in terms of an Euclidean functional integral over all *compact* Riemannian 4-geometries matching the boundary data at  $(\Sigma_2, h_2)$ . Although this approach to quantum cosmology still involves a number of formal definitions, the semiclassical evaluation of the corresponding wave function may be put on solid ground. The 1-loop analysis is related to mathematical and physical subjects such as cobordism theory (i.e., under which conditions a compact manifold is the boundary of another compact manifold), the geometry of compact Riemannian 4-manifolds, the asymptotic heat kernel, the 1-loop effective action, and the use of mixed boundary conditions in quantum field theory (see below).

In particular, over the last decades many efforts have been produced to evaluate 1-loop quantum amplitudes for gauge fields and the gravitational field in the presence of boundaries, either by using the space-time covariant Schwinger-DeWitt method or the mode-by-mode analysis which relies on zeta-function regularization. The main motivations were the need to understand the relation between different approaches to quantum field theories in the presence of boundaries and the quantization of closed cosmologies. Indeed, boundaries play an important role in the Feynman approach to quantum gravity as we just said, in choosing Becchi-Rouet-Stora-Tyutin- (BRST-)

covariant and gauge-invariant boundary conditions for quantum cosmology and in studying different quantization and regularization techniques in field theory. In particular, for the latter problem, discrepancies were found in the semiclassical evaluation of quantum amplitudes by using space-time covariant methods, where the scaling factor of 1-loop quantum amplitudes coincides with the Schwinger-DeWitt  $A_2$  coefficient in the heat-kernel expansion, or instead a mode-by-mode analysis, for which the resulting equations obeyed by the eigenvalues are studied through zeta-function methods [4, 5].

If one reduces a field theory with first-class constraints [6] to its physical degrees of freedom before quantization, one of the main problems is whether the resulting quantum theory is equivalent to the theories relying on the Faddeev-Popov gauge-averaging method or on the extended-phase-space Hamiltonian functional integral of Batalin, Fradkin, and Vilkovisky, where one takes into account ghost and gauge modes. We will see that, in a mode-by-mode evaluation of the covariant functional integral including gauge-averaging and ghost terms, after doing a 3+1 split and a Hodge-like decomposition of the components of metric and ghost perturbations, there are no exact cancellations between contributions of gauge and ghost modes, when linear covariant gauges are used. This lack of cancellation turns out to be essential to achieve agreement between different techniques.

In [7], the  $\zeta(0)$  calculation was performed for gravitons by restricting the functional-integral measure to transverse-traceless perturbations in the case of flat Euclidean 4-space bounded by a 3-sphere. In [8, 9, 10] this result was generalized to the part of the Riemannian de Sitter 4-sphere bounded by a 3-sphere. Both results did not coincide with those obtained by a space-time covariant method [11]. Hence the natural hypothesis arises that the possible non-cancellation of the contributions of gauge and ghost modes can be the cause of the discrepancy. In the work presented in [12, 13] such a suggestion was checked for the electromagnetic field on different manifolds and in different gauges.

In [14] the asymptotic heat kernel for second-order elliptic operators was obtained in the case of pure and mixed boundary conditions in real Riemannian 4-manifolds, and in [15] this analysis was improved. In the light of these results, the conformal anomalies on Einstein spaces with boundaries were recalculated in [16].

In [17] the linearized gravitational field was studied in the geometric framework of [7] (i.e. flat Euclidean 4-space bounded by a 3-sphere), and the resulting  $\zeta(0)$  value was compared with the space-time covariant calcula-

tion of the same Faddeev–Popov amplitudes, by using the corrected geometric formulae for the asymptotic heat kernel in the case of mixed boundary conditions [16].

However, in the case of mixed boundary conditions involving tangential derivatives of metric perturbations, no geometric formulae for the asymptotic heat kernel are available as yet, and one has to resort, to the best of our knowledge, to analytic techniques along the lines of the work in [18, 19]. This is what our review is mainly devoted to. For this purpose, section 2 derives the integral representation of the spectral zeta-functions, as obtained in [8, 20, 21]. Sections 3 and 4 are devoted to  $\zeta(0)$  values for scalar and gauge fields, respectively. Detailed calculations for the gravitational field begin in section 5 and continue until section 7. The strong ellipticity issue is studied in section 8, with examples. Concluding remarks and open problems are presented in section 9.

## 2 Integral representation of the spectral zeta-function

A convenient method for the calculation of the spectral zeta function for the case when the spectrum is not known explicitly, but only the structure of the basis functions of the corresponding differential operator are known, was proposed in [8]. Here we sketch the basic ideas and formulae of this method; for a related approach see Dowker [22, 23].

Let us consider the second-order operator  $F$ , which represents the second functional derivative of the Euclidean action of the model under consideration with respect to the field variables. It is convenient to single out the mass term  $m^2$  from the operator  $F$ . As a manifold we consider the part of the closed Euclidean de Sitter space (“Euclidean ball”). Then suppose that we have a full set of basis functions  $u_A^k(\tau|m^2)$  of this massive operator, i.e.

$$\left[ F_{ik} \left( \frac{d}{d\tau} \right) + m^2 a_{ik} \right] u_A^k(\tau|m^2) = 0. \quad (1)$$

Here  $\tau$  is the Euclidean time parameter, lower case Latin indices  $i$  enumerate the modes of the field variables while capital Latin indices enumerate the basis functions,  $a_{ik}$  are the coefficients of the second-order derivatives with respect to the time parameter.

The only condition which these basis functions should satisfy is regularity in the Euclidean ball  $0 \leq \tau \leq \tau_+$ . Then the eigenvalues  $\lambda$  of the operator  $F(d/d\tau) + m^2 a$  with homogeneous Dirichlet boundary conditions satisfy the equation

$$u_A^i(\tau_+ | m^2 - \lambda) = 0. \quad (2)$$

For other types of boundary conditions the basis functions  $u$  in Eq. (2) should be substituted by the corresponding combination of basis functions and their derivatives. For the case of Dirichlet boundary conditions, which we consider in more detail in this section, the equations defining all the eigenvalues can be collected in one equation,

$$\det u_A^i(\tau_+ | m^2 - \lambda) = 0, \quad (3)$$

in which the determinant is taken with respect to the indices  $i$  and  $A$  of the square matrix  $u_A^i$ . Then using the Cauchy formula and the well-known relation between  $\det$  and  $\text{tr}$ , we can rewrite  $\zeta(s)$  as an integral,

$$\zeta(s) = \frac{1}{2\pi i} \int_C \frac{dz}{z^s} \frac{d}{dz} \text{tr} \ln u(\tau_+ | m^2 - z), \quad (4)$$

over the contour  $C$  in the complex plane of  $z$ , which encircles all roots of (3).

It is necessary to note that positivity and real-valuedness of roots of (3) is guaranteed by self-adjointness and positive definiteness of  $F(d/d\tau) + m^2 a$ , which is assumed here.

It should be stressed also that each basis function may be taken with an overall normalization factor depending on  $(m^2 - z)$ . This can lead to the additional roots of Eq. (3), which are irrelevant to the eigenvalues of the elliptic operator under consideration. To avoid such an effect, consider the singular point of the radial equation at  $\tau = 0$ . The asymptotic behaviour of  $u_A^i(\tau | m^2 - z)$  for  $\tau \rightarrow 0$  has, according to the asymptotic expansion theory, a power-law form [24]

$$u(\tau | m^2 - z) \sim u_0 \tau^k + O(\tau^{k+1}), \quad \tau \rightarrow 0, \quad (5)$$

where  $k$  is a positive integer number. Thus, to avoid the additional roots of Eq. (3), it is sufficient to require that  $u_0$  should be independent of the mass.

If we now assume that the basis functions are analytic in the complex plane of the mass variable  $m^2$ , then we can continuously deform the original contour of integration  $C$  to the new contour  $\tilde{C}$ , which encircles the cut in the

complex plane of the functions  $z^{-s}$ , coinciding with the negative real axis. Thus, the general expression for  $\zeta(s)$ , to be analytically continued to  $s = 0$ , looks like

$$\zeta(s) = \frac{1}{2\pi i} \int_{\tilde{C}} \frac{dz}{z^s} \frac{d}{dz} \text{tr} \ln u(\tau_+ | m^2 - z). \quad (6)$$

For the analytic continuation of (6) from the convergence region domain to  $s = 0$ , take into account that the contour  $\tilde{C}$  includes the two boundaries of the negative real axis and a circle around the point  $z = 0$  of some small radius  $\varepsilon$ . Therefore

$$\begin{aligned} \zeta(s) &= \frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{dM^2}{M^{2s}} \frac{d}{dM^2} \text{tr} \ln u(\tau_+ | m^2 + M^2) \\ &+ \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{dz}{z^s} \frac{d}{dz} \text{tr} \ln u(\tau_+ | m^2 - z), \end{aligned} \quad (7)$$

where the first term is a jump of the integrand in (6) on the cut of the function  $z^{-s}$ , integrated along this cut  $z = -M^2$ .

Let us transform Eq. (7) by the following sequence of operations: first analytically continue both terms into the neighborhood of  $s = 0$  and then go to the limit  $\varepsilon = 0$ . The integral along  $C_\varepsilon$  will vanish because of the regularity of  $u(\tau_+ | m^2 - z)$  at  $z = 0$ .

It is not so difficult to show that for a quantum-mechanical system with a finite number of degrees of freedom, as  $s \rightarrow 0$  we have

$$\zeta(s) = I_{\log} + s[I]_0^\infty + O(s^2), \quad (8)$$

where

$$I(M^2) \equiv \text{tr} \ln u(\tau_+ | m^2 + M^2), \quad (9)$$

$I_{\log}$  is the coefficient of  $\ln M^2$  in the expansion of  $I$  as  $M^2 \rightarrow \infty$ ,  $[I]^\infty$  is the regular part of this expansion as  $M^2 \rightarrow \infty$  and  $[I]_0 = I(0)$ ,  $[I]_0^\infty = [I]^\infty - [I]_0$ . It is obvious that in this case

$$\zeta(0) = I_{\log}, \quad (10)$$

$$\zeta'(0) = [I]_0^\infty. \quad (11)$$

This result shows that  $\zeta(0)$  and  $\zeta'(0)$  get a contribution from the asymptotic value of the basis function  $u_A^i(\tau | M^2)$  for  $M^2 \rightarrow \infty$ . But their asymptotic behaviour can be obtained from the JWKB approximation for the corresponding equation [24].

The problem becomes much more complicated when we study field theories, which have an infinite number of modes, because the trace in (6) becomes the divergent series

$$\zeta(s) = \frac{1}{2\pi i} \int_{\tilde{C}} \frac{dz}{z^s} \frac{d}{dz} \sum_A [\ln u(\tau_+ | m^2 - z)]_A^A. \quad (12)$$

The question arises of how the parameter  $s$  can regularize this divergent series. Let us interchange the summation and integration operations in (12),

$$\zeta(s) = \frac{1}{2\pi i} \sum_A \int_{\tilde{C}} \frac{dz}{z^s} \frac{d}{dz} [\ln u(\tau_+ | m^2 - z)]_A^A, \quad (13)$$

and consider the asymptotic behaviour of the integral

$$\int_{\tilde{C}} \frac{dz}{z^s} \frac{d}{dz} [\ln u(\tau_+ | m^2 - z)]_A^A, \quad (14)$$

for the collective index  $A$  growing to infinity. The numerical parameter tending to infinity with the growth of  $A$  is the parameter  $n$  enumerating the harmonics of the radial equation. Thus, the question of convergence for the sum (13) reduces to the analysis of the asymptotic behaviour of (14) for  $n \rightarrow \infty$ . Fortunately, the so-called uniform JWKB expansion for the basis functions has an important property [24, 25, 8]: when it is considered as a function of the two arguments  $n \rightarrow \infty$  and the ratio  $z/n^2$ ,

$$\ln u(\tau | m^2 - z) = \varphi_{\text{JWKB}} \left( n^2, \frac{z}{n^2} \right), \quad (15)$$

then it is uniform in the second argument,  $0 \leq |z|/n^2 < \infty$ , and at most has a power-law growth of finite order  $k$  in the first argument,  $n^2 \rightarrow \infty$ . Therefore, substituting (15) into (14) and making the change of integration variable  $z \rightarrow n^2 z$  one finds that this integral has an asymptotic behaviour,

$$\frac{1}{n^{2s}} \int_{\tilde{C}} \frac{dz}{z^s} \frac{d}{dz} \varphi_{\text{JWKB}}(n^2, z), \quad (16)$$

which converges for some  $s > 0$  and, due to uniformity of  $\varphi_{\text{JWKB}}(n^2, z)$  in  $z$ , has a bound  $\text{const} \times (n^2)^{k-s}$  providing the convergence of the infinite series (13) for some large positive  $s$ . Thus, large values of  $s > 0$  regularize the divergent sum in (12).

Making the change of integration variable  $z \rightarrow n^2 z$  in Eq. (12) and interchanging back the order of integration and summation one can represent the  $\zeta$  function in the form

$$\zeta(s) = \frac{1}{2\pi i} \int_{\tilde{C}} \frac{dz}{z^s} \frac{d}{dz} I(-z, s), \quad (17)$$

where  $I(-z, s)$  is the manifestly regularized infinite sum

$$I(-z, s) = \sum_A \frac{1}{n^{2s}} [\ln u(\tau_+ | m^2 - z)]_A^A. \quad (18)$$

Similarly to (7), we can split the integral (17) over  $\tilde{C}$  into a sum of two terms and show that the integral around the circle  $C_\varepsilon$  tends to zero. However, unlike models with a finite number of physical variables, the series (18) analytically continued from its convergence domain generally has a pole at  $s = 0$ ,

$$I(M^2, s) = \frac{I^{\text{pole}}(M^2)}{s} + I^R(M^2) + O(s). \quad (19)$$

Therefore, instead of the formula (8) we obtain the following result for the case of field theory:

$$\begin{aligned} \zeta(s) &= (I^R)_{\log} + [I^{\text{pole}}]_0^\infty \\ &+ \left\{ [I^R]_0^\infty - \int_0^\infty dM^2 \ln M^2 \frac{dI^{\text{pole}}(M^2)}{dM^2} \right\} + O(s^2), \end{aligned} \quad (20)$$

where  $(\ )_{\log}$  and  $[\ ]_0^\infty$  have the same sense as in the quantum-mechanical case. Thus we have

$$\zeta(0) = (I^R)_{\log} + [I^{\text{pole}}]_0^\infty, \quad (21)$$

$$\zeta'(0) = [I^R]_0^\infty - \int_0^\infty dM^2 \ln M^2 \frac{dI^{\text{pole}}(M^2)}{dM^2}. \quad (22)$$

These equations generalize the algorithms (10) and (11) to field theories with an infinite number of physical modes. But these generalizations are non-trivial: only the terms  $(I^R)_{\log}$  in (21) and  $[I^R]_0^\infty$  in (22) are similar to the expressions (10) and (11). The terms including  $I^{\text{pole}}$  do not have analogs in a theory with a finite number of modes. These terms are responsible for the non-trivial renormalization of the ultraviolet divergences performed by the  $\zeta$ -function regularization.



The formalism just described is a fine tuned scheme for the computation of  $\zeta(0)$  and  $\zeta'(0)$ . It does, however, not allow to extract other properties of  $\zeta(s)$ . But in order to determine heat kernel coefficients or the Casimir energy associated with some quantum field theory models, other particular properties are needed [26, 27, 28]. These can be found by analytically continuing the zeta function  $\zeta(s)$  as given in Eq. (13) to a meromorphic function in the complex plane. The details of this procedure depend very much on the explicit form of  $u(\tau_+|m^2 - z)$  in Eq. (13). In general one can only say that adding and subtracting the asymptotic expansion briefly outlined in Eq. (15) is crucial to the method, but the precise nature of integrals and series to be done to obtain the analytical continuation depend on exactly what  $\varphi_{JWKB}(n^2, z/n^2)$  actually is. For the example of the scalar Laplacian on a four dimensional ball with various boundary conditions details will be provided in the next section.

### 3 Dirichlet, Neumann and Robin Boundary Conditions

One becomes familiar with Dirichlet boundary conditions as soon as one studies potential theory. The first boundary-value problem of potential theory is the existence of a function, harmonic in a closed region, and taking on preassigned continuous boundary values. This is known as the Dirichlet problem, and is the oldest existence theorem in potential theory. Usually, one first tries to express a harmonic function in terms of its boundary values. One then sees if the expression found continues to represent a harmonic function when the boundary values are any given continuous function.

The problem of finding a function, harmonic in a region, and having normal derivatives equal to the function given on the boundary is instead the Neumann problem, or the second boundary-value problem of potential theory. The theorem asserting the existence of a solution of this problem is known as the second fundamental existence theorem of potential theory [29].

In the semiclassical approximation of the quantum theory of a real scalar field in a real Riemannian background with boundary, the guiding principle for the choice of boundary conditions is that the boundary data should reflect those particular conditions which lead to a well-posed classical boundary-value problem. Thus, on using the background-field method, the scalar-field

perturbations  $\varphi$  are required to obey one of the following three boundary conditions on the bounding surfaces [30]:

(i) Dirichlet problem:

$$\varphi = 0 \quad \text{at} \quad \partial M, \quad (23)$$

(ii) Neumann problem:

$$\frac{\partial \varphi}{\partial \tau} = 0 \quad \text{at} \quad \partial M, \quad (24)$$

(iii) Robin problem:

$$\frac{\partial \varphi}{\partial \tau} + \frac{u}{\tau} \varphi = 0 \quad \text{at} \quad \partial M. \quad (25)$$

For example, in the case of a massless scalar field at 1 loop about flat 4-dimensional Euclidean space bounded by a 3-sphere, the technique of section 2 may be used to find the following values for the resulting anomalous scaling factors:

$$\zeta_D(0) = -\frac{1}{180} \quad (\text{Dirichlet}), \quad (26)$$

$$\zeta_N(0) = \frac{29}{180} \quad (\text{Neumann}), \quad (27)$$

$$\zeta_R(0) = -\frac{1}{180} - \frac{1}{6}(u-1)^3 \quad (\text{Robin}). \quad (28)$$

In this particular case, the  $\zeta(0)$  values coincide with the conformal anomaly, since massless scalar field theories are conformally invariant in flat space-time. It was not until in [31] that a powerful *analytic* algorithm was developed for the analysis of the Robin case, and the first correct *geometric* results for  $\zeta(0)$  were only published in [32] and [15, 16]. More recent work on real scalar fields on the Euclidean ball in various dimensions can be found in [20, 21, 22, 23, 33].

In order to outline the contour integration method for the analysis of spectral zeta functions as a function of  $s$  let us exploit this opportunity and rederive Eq. (26) with an indication on how to obtain Eqs. (27)-(28). Given the treatment of arbitrary dimension does not cause any additional complications, we will consider the  $D = d + 1$  dimensional ball [26].

A massless scalar field leads to the eigenvalue problem for a Laplacian, and for the spherically symmetric problem at hand the use of polar coordinates

seems appropriate. In these coordinates the Laplacian reads

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\mathcal{N}}, \quad (29)$$

with  $\Delta_{\mathcal{N}}$  the Laplacian on the  $d$ -dimensional sphere,  $\mathcal{N} = S^d$ .

By imposing Dirichlet boundary conditions on the sphere, the boundary of the ball, eigenvalues are determined by the transcendental equation

$$J_{\nu}(\lambda) = 0, \quad (30)$$

with  $\nu = \ell + (d-1)/2$ ,  $\ell = 0, 1, 2, \dots$ , and with the radius  $R$  of the ball being chosen as  $R = 1$ . The degeneracy  $d_{\nu}$  of each eigenvalue equals the degeneracy of the eigenvalues of the Laplacian on the  $d$ -sphere and for  $d \geq 2$ , the case we will concentrate on in the following, it equals

$$d_{\nu} = (2\ell + d - 1) \frac{(\ell + d - 2)!}{\ell!(d-1)!}. \quad (31)$$

The information provided suffices to give an explicit representation of the associated zeta function as given in Eq. (13), i.e.

$$\begin{aligned} \zeta(s) &= \frac{1}{2\pi i} \sum_{\nu} d_{\nu} \int_{\tilde{C}} \frac{dz}{z^{2s}} \frac{d}{dz} \ln(z^{-\nu} J_{\nu}(z)) \\ &= \frac{\sin \pi s}{\pi} \sum_{\nu} d_{\nu} \int_0^{\infty} \frac{dk}{k^{2s}} \frac{d}{dk} \ln(k^{-\nu} I_{\nu}(k)), \end{aligned} \quad (32)$$

where this equality is obtained by deforming the contour  $\tilde{C}$  to the imaginary axis. The relevant uniform asymptotic behaviour outlined in Eq. (15) for the given example follows from

$$I_{\nu}(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right], \quad (33)$$

valid for  $\nu \rightarrow \infty$  as  $z = k/\nu$  is fixed [24, 34]. Here  $t = 1/\sqrt{1+z^2}$  and  $\eta = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$ . Higher powers in  $t$  follow from the recursion [34]

$$u_{k+1}(t) = \frac{1}{2} t^2 (1-t^2) u'_k(t) + \frac{1}{8} \int_0^t d\tau (1-5\tau^2) u_k(\tau),$$

starting with  $u_0(t) = 1$ . On defining polynomials  $D_n(t)$  from the expansion

$$\ln \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right] \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n}, \quad (34)$$

the leading few polynomials are

$$\begin{aligned} D_1(t) &= \frac{1}{8}t - \frac{5}{24}t^3, \\ D_2(t) &= \frac{1}{16}t^2 - \frac{3}{8}t^4 + \frac{5}{16}t^6, \\ D_3(t) &= \frac{25}{384}t^3 - \frac{531}{640}t^5 + \frac{221}{128}t^7 - \frac{1105}{1152}t^9, \end{aligned} \quad (35)$$

with many more polynomials easily found by using an algebraic computer program.

By adding and subtracting  $N$  leading terms, for  $\zeta(0)$  in  $D = 4$  we will ultimately choose  $N = 3$ , the zeta function (32) splits into the pieces (after substituting  $k = z\nu$ )

$$\zeta(s) = Z(s) + \sum_{i=-1}^N A_i(s),$$

where

$$\begin{aligned} Z(s) &= \frac{\sin(\pi s)}{\pi} \sum_{\nu} d_{\nu} \int_0^{\infty} dz (z\nu)^{-2s} \frac{\partial}{\partial z} \left\{ \ln [z^{-\nu} I_{\nu}(z\nu)] \right. \\ &\quad \left. - \ln \left[ \frac{z^{-\nu}}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \right] - \sum_{n=1}^N \frac{D_n(t)}{\nu^n} \right\}, \end{aligned} \quad (36)$$

and the  $A_i(s)$  result from the different orders in the asymptotic expansion, explicitly

$$\begin{aligned} A_{-1}(s) &= \frac{\sin(\pi s)}{\pi} \sum_{\nu} d_{\nu} \int_0^{\infty} dz (z\nu)^{-2s} \frac{\partial}{\partial z} \ln (z^{-\nu} e^{\nu\eta}), \\ A_0(s) &= \frac{\sin(\pi s)}{\pi} \sum_{\nu} d_{\nu} \int_0^{\infty} dz (z\nu)^{-2s} \frac{\partial}{\partial z} \ln(1+z^2)^{-1/4}, \\ A_i(s) &= \frac{\sin(\pi s)}{\pi} \sum_{\nu} d_{\nu} \int_0^{\infty} dz (z\nu)^{-2s} \frac{\partial}{\partial z} \left( \frac{D_i(t)}{\nu^i} \right). \end{aligned}$$

It can be shown that  $Z(s)$  is analytic in the half-plane  $(d-1-N)/2 < \Re s$ . These formulas therefore make it possible to find a representation of  $\zeta(s)$  valid for any value of  $s$ .

Choosing  $N$  suitably large, given the factor  $\sin(\pi s)$  in (36),  $Z(s)$  will therefore not contribute to  $\zeta(0)$ . As far as  $\zeta(0)$  is concerned, it therefore suffices to only consider the  $A_i(s)$  further. By introducing the so-called base zeta function,

$$\zeta_{\mathcal{N}}(s) = \sum_{\nu} d_{\nu} \nu^{-2s}, \quad (37)$$

$A_{-1}(s)$  and  $A_0(s)$  are readily evaluated as

$$\begin{aligned} A_{-1}(s) &= \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s+1)} \zeta_{\mathcal{N}}\left(s - \frac{1}{2}\right), \\ A_0(s) &= -\frac{1}{4} \zeta_{\mathcal{N}}(s). \end{aligned}$$

In order to compute the higher  $A_i(s)$  note that the polynomials  $D_i(t)$  can be written as

$$D_i(t) = \sum_{b=0}^i x_{i,b} t^{i+2b},$$

with the coefficients  $x_{i,b}$  easily determined from the definition (34) of  $D_i(t)$ ; to read off the numbers  $x_{i,b}$  for  $i = 1, 2, 3$ , see also (35). The  $z$ -integrals are then easily done and one finds

$$A_i(s) = -\frac{1}{\Gamma(s)} \zeta_{\mathcal{N}}\left(s + \frac{i}{2}\right) \sum_{b=0}^i x_{i,b} \frac{\Gamma(s+b+\frac{i}{2})}{\Gamma(b+\frac{i}{2})}.$$

Concentrating on  $\zeta(0)$  in four dimensions we note that

$$\zeta_{\mathcal{N}}(s) = \sum_{\ell=0}^{\infty} (\ell+1)^2 (\ell+1)^{-2s} = \zeta_R(2s-2),$$

which allows us to write

$$\begin{aligned}
A_{-1}(s) &= \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s + 1)} \zeta_R(2s - 3), \\
A_0(s) &= -\frac{1}{4} \zeta_R(2s - 2), \\
A_i(s) &= -\frac{1}{\Gamma(s)} \zeta_R(2s + i - 2) \sum_{b=0}^i x_{i,b} \frac{\Gamma(s + b + \frac{i}{2})}{\Gamma(b + \frac{i}{2})}.
\end{aligned}$$

At  $s = 0$  we compute

$$\begin{aligned}
A_{-1}(0) &= \frac{1}{4\sqrt{\pi}} \Gamma\left(-\frac{1}{2}\right) \zeta_R(-3) = -\frac{1}{240}, \\
A_0(0) &= A_1(0) = A_2(0) = 0, \\
A_3(0) &= -\frac{1}{2} \sum_{b=0}^3 x_{3,b} = -\frac{1}{720},
\end{aligned}$$

and thus as stated

$$\zeta(0) = -\frac{1}{240} - \frac{1}{720} = -\frac{1}{180}.$$

In the same manner Neumann and Robin boundary conditions can be treated by starting with the implicit eigenvalue equation

$$\left(1 - \frac{D}{2} + u\right) J_\nu(\lambda) + \lambda J'_\nu(\lambda) = 0,$$

where  $u = 0$  corresponds to Neumann boundary conditions; for details about the very similar calculations we refer to [26].

For complex scalar fields, the most general case corresponds to mixed boundary conditions, i.e. when the real part obeys Dirichlet conditions and the imaginary part obeys Neumann conditions, or the other way around. In the light of (26)–(27), the resulting conformal anomaly for a complex massless field on the Euclidean ball is found to be

$$\zeta(0) = \frac{7}{45}. \tag{38}$$

## 4 Mixed boundary conditions for gauge fields

We are interested in the 1-loop amplitudes of vacuum Maxwell theory in the presence of boundaries. Since in the classical theory the potential  $A_\mu$  is subject to the gauge transformations

$$\hat{A}_\mu \equiv A_\mu + \partial_\mu \varphi, \quad (39)$$

this gauge freedom is reflected in the quantum theory by a ghost 0-form, i.e. an anticommuting, complex scalar field, hereafter denoted again by  $\varphi$ . The two sets of mixed boundary conditions consistent with gauge invariance and Becchi–Rouet–Stora–Tyutin (hereafter BRST) symmetry are magnetic, i.e.

$$\left[ A_k \right]_{\partial M} = 0, \quad (40)$$

$$\left[ \Phi(A) \right]_{\partial M} = 0, \quad (41)$$

$$\left[ \varphi \right]_{\partial M} = 0, \quad (42)$$

or electric, i.e.

$$\left[ A_0 \right]_{\partial M} = 0, \quad (43)$$

$$\left[ \frac{\partial A_k}{\partial \tau} \right]_{\partial M} = 0, \quad (44)$$

$$\left[ \frac{\partial \varphi}{\partial \tau} \right]_{\partial M} = 0, \quad (45)$$

where  $\Phi$  is an arbitrary gauge-averaging functional defined on the space of connection 1-forms  $A_\mu dx^\mu$ . Note that the boundary condition (42) ensures the gauge invariance of the boundary conditions (40)-(41) on making the gauge transformation (39). Similarly, the boundary condition (45) ensures the gauge invariance of (43)-(44) on transforming the potential as in (39). For example, when the Lorenz gauge-averaging functional is chosen,

$$\Phi_L(A) \equiv \nabla^\mu A_\mu,$$

the boundary condition (41) reduces to ( $K$  being the extrinsic-curvature tensor of the boundary)

$$\left[ \frac{\partial A_0}{\partial \tau} + A_0 \text{Tr } K \right]_{\partial M} = 0,$$

by virtue of Eq. (40) that sets to zero at the boundary all longitudinal and transverse modes.

It is also instructive to prove the BRST invariance of our boundary conditions. For this purpose, take e.g. the electric boundary conditions (43)–(45), jointly with the BRST transformations (the ghost 0-form corresponding to independent and real-valued ghost fields denoted by  $\omega$  and  $\psi$ , while  $\delta\lambda$  is an anti-commuting gauge parameter)

$$\delta_{\text{BRST}} A_\mu = (\nabla_\mu \psi) \delta\lambda, \quad (46)$$

$$\delta_{\text{BRST}} \omega = (\nabla^\mu A_\mu) \delta\lambda, \quad (47)$$

$$\delta_{\text{BRST}} \psi = 0. \quad (48)$$

Now if  $\psi$  obeys Neumann boundary conditions,

$$\left[ n_\mu \nabla^\mu \psi \right]_{\partial M} = 0, \quad (49)$$

then by virtue of (46) one finds (hereafter  $\widehat{\delta} \equiv \delta_{\text{BRST}}$ )

$$\widehat{\delta}(n_\mu A^\mu) = n_\mu (\widehat{\delta} A^\mu) = (\delta\lambda) n_\mu (\nabla^\mu \psi), \quad (50)$$

and this variation vanishes at the boundary by virtue of (49). Thus, the boundary condition

$$\left[ n_\mu A^\mu \right]_{\partial M} = 0, \quad (51)$$

which is the covariant form of (43), is preserved under the action of BRST transformations. Further details can be found in [11].

For a given choice of one of these two sets of mixed boundary conditions, different choices of background 4-geometry, boundary 3-geometry and gauge-averaging functional lead to a number of interesting results. We here summarize them in the case of a background given by flat Euclidean 4-space bounded by one 3-sphere (i.e. the disk) or by two concentric 3-spheres (i.e. the ring).

(i) The operator matrix acting on normal and longitudinal modes of the potential can be put in diagonal form for all relativistic gauge conditions which can be expressed as

$$\Phi_b(A) \equiv \nabla^\mu A_\mu - b A_0 \text{Tr } K, \quad (52)$$



where  $\nabla^\mu$  denotes covariant differentiation with respect to the Levi-Civita connection of the background, and  $b$  is a dimensionless parameter.

(ii) In the case of the disk, the Lorenz gauge (set  $b = 0$  in (52)) leads to a  $\zeta(0)$  value

$$\zeta_L(0) = -\frac{31}{90}, \quad (53)$$

for both magnetic and electric boundary conditions, which agrees [16] with the geometric theory of the asymptotic heat kernel. However, the  $\zeta(0)$  value depends on the gauge condition, and unless  $b$  vanishes it also depends on the boundary conditions.

(iii) In the case of the ring, one finds

$$\zeta(0) = 0, \quad (54)$$

for all gauge conditions, independently of boundary conditions. This result agrees with the geometric formulae for the heat kernel, since volume (i.e. interior) contributions to  $\zeta(0)$  vanish in a flat background, while surface (i.e. boundary) contributions cancel each other.

(iv) In the case of boundary 3-geometries given by one or two 3-spheres, the most general gauge-averaging functional takes the form [13, 35]

$$\Phi(A) = \gamma_1 {}^{(4)}\nabla^0 A_0 + \frac{\gamma_2}{3} A_0 \text{Tr } K - \gamma_3 {}^{(3)}\nabla^i A_i, \quad (55)$$

where  $\gamma_1, \gamma_2$  and  $\gamma_3$  are arbitrary dimensionless parameters, which give different “weight” to the various terms in the 3+1 decomposition of  $\nabla^\mu A_\mu$ . Thus, unless  $\gamma_1, \gamma_2$  and  $\gamma_3$  take some special values (cf. (52)), it is not possible to diagonalize the operator matrix acting on normal and longitudinal modes of the potential.

(v) The contributions to  $\zeta(0)$  resulting from normal and longitudinal modes *do not* cancel in general the contribution of ghost modes. Thus, *transverse modes do not provide the only surviving contribution to 1-loop amplitudes*. In other words, *all* perturbative modes are necessary to recover the correct form of 1-loop semiclassical amplitudes.

## 5 Boundary conditions for the gravitational field

For gauge fields and gravitation, the boundary conditions are mixed in that some components of the field (more precisely, a 1-form or a symmetric tensor of type  $(0, 2)$ ) obey a set of boundary conditions, and the remaining part of the field obeys another set of boundary conditions. Moreover, the boundary conditions are invariant under local gauge transformations provided that suitable boundary conditions are imposed on the corresponding ghost 0-form or 1-form.

We are here interested in the derivation of mixed boundary conditions for Euclidean quantum gravity. The knowledge of the classical variational problem, and the principle of gauge invariance, are enough to lead to a highly non-trivial quantum boundary-value problem. Indeed, it is by now well-known that, if one fixes the 3-metric at the boundary in general relativity, the corresponding variational problem is well-posed and leads to the Einstein equations, providing the Einstein-Hilbert action is supplemented by a boundary term whose integrand is proportional to the trace of the second fundamental form. In the corresponding quantum boundary-value problem, which is relevant for the 1-loop approximation in quantum gravity, the perturbations  $h_{ij}$  of the induced 3-metric are set to zero at the boundary. Moreover, the whole set of metric perturbations  $h_{\mu\nu}$  are subject to the so-called infinitesimal *gauge* transformations

$$\hat{h}_{\mu\nu} \equiv h_{\mu\nu} + \nabla_{(\mu} \varphi_{\nu)}, \quad (56)$$

where  $\nabla$  is the Levi-Civita connection of the background 4-geometry with metric  $g$ , and  $\varphi_\nu dx^\nu$  is the ghost 1-form. In geometric language, the difference between  $\hat{h}_{\mu\nu}$  and  $h_{\mu\nu}$  is given by the Lie derivative along  $\varphi$  of the 4-metric  $g$ .

For problems with boundary, Eq. (56) implies that

$$\hat{h}_{ij} = h_{ij} + \varphi_{(i|j)} + K_{ij}\varphi_0, \quad (57)$$

where the stroke denotes, as usual, 3-dimensional covariant differentiation tangentially with respect to the intrinsic Levi-Civita connection of the boundary, while  $K_{ij}$  is the extrinsic-curvature tensor of the boundary. Of course,  $\varphi_0$  and  $\varphi_i$  are the normal and tangential components of the ghost 1-form, respectively. Note that boundaries make it necessary to perform a 3+1 split

of space-time geometry and physical fields. As such, they introduce non-covariant elements in the analysis of problems relevant for quantum gravity. This seems to be an unavoidable feature, although the boundary conditions may be written in a covariant way.

In light of (57), the boundary conditions

$$\left[ h_{ij} \right]_{\partial M} = 0 \quad (58)$$

are gauge invariant, i.e.

$$\left[ \widehat{h}_{ij} \right]_{\partial M} = 0, \quad (59)$$

if and only if the whole ghost 1-form obeys homogeneous Dirichlet conditions, so that

$$\left[ \varphi_0 \right]_{\partial M} = 0, \quad (60)$$

$$\left[ \varphi_i \right]_{\partial M} = 0. \quad (61)$$

The conditions (60) and (61) are necessary and sufficient since  $\varphi_0$  and  $\varphi_i$  are independent, and 3-dimensional covariant differentiation commutes with the operation of restriction at the boundary. Indeed, we are assuming that the boundary is smooth and not totally geodesic, i.e.  $K_{ij} \neq 0$ . However, *at those points of  $\partial M$  where the extrinsic-curvature tensor vanishes, the condition (60) is no longer necessary* [35].

The problem now arises to impose boundary conditions on the remaining set of metric perturbations. The key point is to make sure that the invariance of such boundary conditions under the infinitesimal transformations (56) is again guaranteed by (60)-(61), since otherwise one would obtain incompatible sets of boundary conditions on the ghost 1-form. Indeed, on using the Faddeev–Popov formalism for the amplitudes of quantum gravity, it is necessary to use a gauge-averaging term in the Euclidean action, of the form

$$I_{\text{g.a.}} \equiv \frac{1}{32\pi G\alpha} \int_M \Phi^\mu g_{\mu\nu} \Phi^\nu \sqrt{\det g} \, d^4x, \quad (62)$$

where  $\Phi^\mu$  is any relativistic gauge-averaging functional which leads to self-adjoint elliptic operators on metric and ghost perturbations. One then finds that (here  $\Phi_\mu \equiv g_{\mu\nu} \Phi^\nu$ )

$$\delta\Phi_\mu(h) \equiv \Phi_\mu(h) - \Phi_\mu(\widehat{h}) = \mathcal{F}_\mu{}^\nu \varphi_\nu, \quad (63)$$

where  $\mathcal{F}_\mu{}^\nu$  is an elliptic operator that acts linearly on the ghost 1-form. Thus, if one imposes the boundary conditions

$$\left[\Phi_0(h)\right]_{\partial M} = 0, \quad (64)$$

$$\left[\Phi_i(h)\right]_{\partial M} = 0, \quad (65)$$

their invariance under (56) is guaranteed when (60) and (61) hold, by virtue of (63). Hence one also has

$$\left[\Phi_0(\widehat{h})\right]_{\partial M} = 0, \quad (66)$$

$$\left[\Phi_i(\widehat{h})\right]_{\partial M} = 0. \quad (67)$$

In section 7 we shall study this scheme, first proposed in [36], when the linear covariant gauge of the de Donder type is chosen. We will see that this leads to boundary conditions which involve normal and tangential derivatives of normal components of metric perturbations, and the resulting 1-loop divergence will be evaluated.

## 6 Equations for basis functions and their solutions for pure gravity

For the reasons described in the introduction, we study pure gravity at 1 loop about a flat Euclidean background with two concentric 3-sphere boundaries, and eventually let one of the 3-spheres shrink to a point. Our approach to quantization follows the Feynman–DeWitt–Faddeev–Popov formalism [37]. Hence we deal with quantum amplitudes of the form

$$Z[\text{boundary data}] = \int_C \mu_1[g] \mu_2[\varphi] \exp(-\widetilde{I}_E).$$

With our notation,  $C$  is the set of all Riemannian 4-geometries matching the boundary data,  $\mu_1$  is a suitable measure on the space of metrics,  $\mu_2$  is a suitable measure for ghosts,  $\Phi^\mu$  is an arbitrary gauge-averaging functional, and the total Euclidean action reads (in  $c = 1$  units)

$$\begin{aligned} \widetilde{I}_E &= I_{\text{gh}} + \frac{1}{16\pi G} \int_M {}^{(4)}R \sqrt{\det g} \, d^4x + \frac{1}{8\pi G} \int_{\partial M} \text{Tr} K \sqrt{\det q} \, d^3x \\ &+ \frac{1}{16\pi G} \int_M \frac{1}{2\alpha} \Phi^\mu g_{\mu\nu} \Phi^\nu \sqrt{\det g} \, d^4x, \end{aligned} \quad (68)$$

where  ${}^{(4)}R$  is the trace of the 4-dimensional Ricci tensor. Of course,  $K$  is the extrinsic-curvature tensor of the boundary,  $q$  is the induced 3-metric of  $\partial M$ , and  $\alpha$  is a positive dimensionless parameter. The ghost action  $I_{\text{gh}}$  depends on the specific form of  $\Phi^\mu$ . Denoting by  $h_{\mu\nu}$  the perturbation about the background 4-metric  $g_{\mu\nu}$ , one thus finds field equations of the kind

$$\square^\Phi h_{\mu\nu} = 0,$$

where  $\square^\Phi$  is the 4-dimensional elliptic operator corresponding to the form of  $\Phi_\nu \equiv g_{\nu\mu}\Phi^\mu$  one is working with. Here we choose the de Donder gauge-averaging functional

$$\Phi_\nu^{DD} \equiv \nabla^\mu \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{h} \right),$$

where  $\nabla^\mu$  is covariant differentiation with respect to  $g_{\mu\nu}$ , and  $\hat{h} \equiv g^{\mu\nu} h_{\mu\nu}$ . The corresponding  $\square^{DD}$  operator is the one obtained by analytic continuation of the standard D'Alembert operator, hereafter denoted by  $\square$ . The resulting eigenvalue equation is

$$\square h_{\mu\nu}^{(\lambda)} + \lambda h_{\mu\nu}^{(\lambda)} = 0.$$

Now we can make the 3+1 decomposition of our background 4-geometry and expand  $h_{00}$ ,  $h_{0i}$  and  $h_{ij}$  in hyperspherical harmonics as

$$h_{00}(x, \tau) = \sum_{n=1}^{\infty} a_n(\tau) Q^{(n)}(x), \quad (69)$$

$$h_{0i}(x, \tau) = \sum_{n=2}^{\infty} \left[ b_n(\tau) \frac{Q_{|i}^{(n)}(x)}{(n^2 - 1)} + c_n(\tau) S_i^{(n)}(x) \right], \quad (70)$$

$$\begin{aligned} h_{ij}(x, \tau) &= \sum_{n=3}^{\infty} d_n(\tau) \left[ \frac{Q_{|ij}^{(n)}(x)}{(n^2 - 1)} + \frac{c_{ij}}{3} Q^{(n)}(x) \right] + \sum_{n=1}^{\infty} \frac{e_n(\tau)}{3} c_{ij} Q^{(n)}(x) \\ &+ \sum_{n=3}^{\infty} \left[ f_n(\tau) \left( S_{i|j}^{(n)}(x) + S_{j|i}^{(n)}(x) \right) + k_n(\tau) G_{ij}^{(n)}(x) \right]. \end{aligned} \quad (71)$$

Here  $Q^{(n)}(x)$ ,  $S_i^{(n)}(x)$  and  $G_{ij}^{(n)}(x)$  are scalar, transverse vector and transverse-traceless tensor hyperspherical harmonics, respectively, on a unit 3-sphere with metric  $c_{ij}$ .

The insertion of the expansions (69)–(71) into Eq. (68) leads to the following system of equations,

$$\hat{A}_n a_n(\tau) + \hat{B}_n b_n(\tau) + \hat{C}_n e_n(\tau) = 0, \quad (72)$$

$$\hat{D}_n b_n(\tau) + \hat{E}_n a_n(\tau) + \hat{F}_n d_n(\tau) + \hat{G}_n e_n(\tau) = 0, \quad (73)$$

$$\hat{L}_n d_n(\tau) + \hat{M}_n b_n(\tau) = 0, \quad (74)$$

$$\hat{N}_n e_n(\tau) + \hat{P}_n b_n(\tau) + \hat{Q}_n a_n(\tau) = 0, \quad (75)$$

$$\hat{H}_n c_n(\tau) + \hat{K}_n f_n(\tau) = 0, \quad (76)$$

$$\hat{R}_n f_n(\tau) + \hat{S}_n c_n(\tau) = 0, \quad (77)$$

$$\hat{T}_n k_n(\tau) = 0. \quad (78)$$

Since our background is flat, after setting  $\alpha = 1$  in (68) the operators appearing in Eqs. (72)–(78) take the form (for all integer  $n \geq 3$ )

$$\hat{A}_n \equiv \frac{d^2}{d\tau^2} + \frac{3}{\tau} \frac{d}{d\tau} - \frac{(n^2 + 5)}{\tau^2} + \lambda_n, \quad (79)$$

$$\hat{B}_n \equiv \frac{4}{\tau^3}, \quad (80)$$

$$\hat{C}_n \equiv \frac{2}{\tau^4}, \quad (81)$$

$$\hat{D}_n \equiv \frac{d^2}{d\tau^2} + \frac{1}{\tau} \frac{d}{d\tau} - \frac{(n^2 + 4)}{\tau^2} + \lambda_n, \quad (82)$$

$$\hat{E}_n \equiv \frac{2}{\tau}(n^2 - 1), \quad (83)$$

$$\hat{F}_n \equiv \frac{4}{3} \frac{(n^2 - 4)}{\tau^3}, \quad (84)$$

$$\hat{G}_n \equiv -\frac{2}{3} \frac{(n^2 - 1)}{\tau^3}, \quad (85)$$

$$\hat{H}_n \equiv \frac{d^2}{d\tau^2} + \frac{1}{\tau} \frac{d}{d\tau} - \frac{(n^2 + 5)}{\tau^2} + \lambda_n, \quad (86)$$

$$\hat{K}_n \equiv \frac{2}{\tau^3}(n^2 - 4), \quad (87)$$

$$\hat{L}_n \equiv \frac{d^2}{d\tau^2} - \frac{1}{\tau} \frac{d}{d\tau} - \frac{(n^2 - 5)}{\tau^2} + \lambda_n, \quad (88)$$

$$\widehat{M}_n \equiv \frac{4}{\tau}, \quad (89)$$

$$\widehat{N}_n \equiv \frac{d^2}{d\tau^2} - \frac{1}{\tau} \frac{d}{d\tau} - \frac{(n^2 + 1)}{\tau^2} + \lambda_n, \quad (90)$$

$$\widehat{P}_n \equiv -\frac{4}{\tau}, \quad (91)$$

$$\widehat{Q}_n \equiv 6, \quad (92)$$

$$\widehat{R}_n \equiv \frac{d^2}{d\tau^2} - \frac{1}{\tau} \frac{d}{d\tau} - \frac{(n^2 - 4)}{\tau^2} + \lambda_n, \quad (93)$$

$$\widehat{S}_n \equiv \frac{2}{\tau}, \quad (94)$$

$$\widehat{T}_n \equiv \frac{d^2}{d\tau^2} - \frac{1}{\tau} \frac{d}{d\tau} - \frac{(n^2 - 1)}{\tau^2} + \lambda_n. \quad (95)$$

Inserting the operator  $\widehat{T}_n$  from Eq. (95) into Eq. (78) one can easily find the basis function describing the transverse-traceless symmetric tensor harmonics which usually are treated as physical degrees of freedom:

$$k_n(\tau) = \alpha_1 \tau I_n(M\tau) + \alpha_2 \tau K_n(M\tau), \quad n = 3, \dots, \quad (96)$$

where  $M = \sqrt{-\lambda}$  and  $I$  and  $K$  are modified Bessel functions.

However, the equations (72)–(75) for scalar-type gravitational perturbations lead to a rather complicated entangled system as well as Eqs. (76) and (77), describing vector perturbations. In Refs. [12, 13], where the analogous problem was studied for the electromagnetic field, a method was used to decouple a similar entangled system for normal and longitudinal components of the 4-vector potential. The idea is that one can diagonalize a  $2 \times 2$  operator matrix after multiplying it by two functional matrices. In some cases one can choose these functional matrices in such a way that the transformed operator matrix is diagonal and the corresponding differential equations for basis functions are decoupled. However, in the case of scalar-type gravitational perturbations we have a  $4 \times 4$  operator matrix. To diagonalize such a matrix it is necessary to solve a system of 24 second-order algebraic equations with 24 variables. This problem seems a rather cumbersome one and we thus use another method. For this purpose, we assume that the solution of the system of equations (72)–(75) is some set of modified Bessel functions with unknown index  $\nu$ . Let us look for a solution of this system in the form

$$a_n(\tau) = \beta_1 \frac{W_\nu(M\tau)}{\tau}, \quad (97)$$

$$b_n(\tau) = \beta_2 W_\nu(M\tau), \quad (98)$$

$$d_n(\tau) = \beta_3 \tau W_\nu(M\tau), \quad (99)$$

$$e_n(\tau) = \beta_4 \tau W_\nu(M\tau). \quad (100)$$

Here,  $W_\nu$  is a linear combination of modified Bessel functions  $I_\nu$  and  $K_\nu$  obeying the Bessel equation

$$\left( \frac{d^2}{d\tau^2} + \frac{1}{\tau} \frac{d}{d\tau} - \frac{\nu^2}{\tau^2} - M^2 \right) W_\nu(M\tau) = 0. \quad (101)$$

Now, inserting the functions (97)–(100) and the corresponding operators into the system of equations (72)–(75), and taking into account the Bessel equation (101), one finds the following system of equations for  $\beta_1, \beta_2, \beta_3$  and  $\beta_4$ ,

$$(\nu^2 - n^2 - 6)\beta_1 + 4\beta_2 + 2\beta_4 = 0, \quad (102)$$

$$6(n^2 - 1)\beta_1 + 3(\nu^2 - n^2 - 4)\beta_2 + 4(n^2 - 4)\beta_3 - 2(n^2 - 1)\beta_4 = 0, \quad (103)$$

$$4\beta_2 + (\nu^2 - n^2 + 4)\beta_3 = 0, \quad (104)$$

$$6\beta_1 - 4\beta_2 + (\nu^2 - n^2 - 2)\beta_4 = 0. \quad (105)$$

The condition for the existence of non-trivial solutions of the linear homogeneous system (102)–(105) is the vanishing of its determinant, i.e.

$$(\nu^2 - n^2)^2 \left[ (\nu^2 - n^2)^2 - 8(\nu^2 - n^2) - 16(n^2 - 1) \right] = 0. \quad (106)$$

The roots of Eq. (106) are

$$\nu^2 = n^2, \quad \nu^2 = (n - 2)^2, \quad \nu^2 = (n + 2)^2.$$

The positive values of  $\nu$  provide the orders of modified Bessel functions. Now we can write down the  $\beta$ 's corresponding to different values for  $\nu$ 's. For  $\nu = n$  one has

$$\beta_4 = 3\beta_1, \quad \beta_2 = \beta_3 = 0, \quad (107)$$

or

$$\beta_1 = 0, \quad \beta_3 = -\beta_2, \quad \beta_4 = -2\beta_2. \quad (108)$$

For  $\nu = n - 2$  one has

$$\beta_2 = (n + 1)\beta_1, \quad \beta_3 = \frac{(n + 1)}{(n - 2)}\beta_1, \quad \beta_4 = -\beta_1. \quad (109)$$



Last, for  $\nu = n + 2$  one has

$$\beta_2 = -(n-1)\beta_1, \quad \beta_3 = \frac{(n-1)}{(n+2)}\beta_1, \quad \beta_4 = -\beta_1. \quad (110)$$

Having the Eqs. (107)–(110) we can get the basis functions for scalar-type gravitational perturbations (97)–(100),

$$\begin{aligned} a_n(\tau) &= \frac{1}{\tau} \left[ \gamma_1 I_n(M\tau) + \gamma_3 I_{n-2}(M\tau) + \gamma_4 I_{n+2}(M\tau) \right. \\ &\quad \left. + \delta_1 K_n(M\tau) + \delta_3 K_{n-2}(M\tau) + \delta_4 K_{n+2}(M\tau) \right], \end{aligned} \quad (111)$$

$$\begin{aligned} b_n(\tau) &= \gamma_2 I_n(M\tau) + (n+1)\gamma_3 I_{n-2}(M\tau) \\ &\quad - (n-1)\gamma_4 I_{n+2}(M\tau) + \delta_2 K_n(M\tau) \\ &\quad + (n+1)\delta_3 K_{n-2}(M\tau) - (n-1)\delta_4 K_{n+2}(M\tau), \end{aligned} \quad (112)$$

$$\begin{aligned} d_n(\tau) &= \tau \left[ -\gamma_2 I_n(M\tau) + \frac{(n+1)}{(n-2)}\gamma_3 I_{n-2}(M\tau) \right. \\ &\quad + \frac{(n-1)}{(n+2)}\gamma_4 I_{n+2}(M\tau) - \delta_2 K_n(M\tau) \\ &\quad \left. + \frac{(n+1)}{(n-2)}\delta_3 K_{n-2}(M\tau) + \frac{(n-1)}{(n+2)}\delta_4 K_{n+2}(M\tau) \right], \end{aligned} \quad (113)$$

$$\begin{aligned} e_n(\tau) &= \tau \left[ 3\gamma_1 I_n(M\tau) - 2\gamma_2 I_n(M\tau) - \gamma_3 I_{n-2}(M\tau) \right. \\ &\quad - \gamma_4 I_{n+2}(M\tau) + 3\delta_1 K_n(M\tau) - 2\delta_2 K_n(M\tau) \\ &\quad \left. - \delta_3 K_{n-2}(M\tau) - \delta_4 K_{n+2}(M\tau) \right]. \end{aligned} \quad (114)$$

We can find the basis functions for vectorlike gravitational perturbations in a similar way. Let us suppose that

$$c_n(\tau) = \varepsilon_1 W_\nu(M\tau), \quad (115)$$

and

$$f_n(\tau) = \varepsilon_2 \tau W_\nu(M\tau). \quad (116)$$

Inserting (115) and (116) into Eqs. (76) and (77) one has the system

$$\begin{aligned}(\nu^2 - n^2 - 5)\varepsilon_1 + 2(n^2 - 4)\varepsilon_2 &= 0, \\ 2\varepsilon_1 + (\nu^2 - n^2 + 3)\varepsilon_2 &= 0.\end{aligned}\tag{117}$$

The determinant of the system (117) is

$$(\nu^2 - n^2)^2 - 2(\nu^2 - n^2) - 4n^2 + 1,\tag{118}$$

and its positive roots are  $n \pm 1$ . For  $\nu = n + 1$  one has

$$\varepsilon_2 = -\frac{1}{(n+2)}\varepsilon_1,$$

and for  $\nu = n - 1$  one has

$$\varepsilon_2 = \frac{1}{(n-2)}\varepsilon_1,$$

and correspondingly the basis functions (115) and (116) take the form

$$c_n(\tau) = \tilde{\varepsilon}_1 I_{n+1}(M\tau) + \tilde{\varepsilon}_2 I_{n-1}(M\tau) + \eta_1 K_{n+1}(M\tau) + \eta_2 K_{n-1}(M\tau),\tag{119}$$

$$\begin{aligned}f_n(\tau) &= \tau \left[ -\frac{1}{(n+2)}\tilde{\varepsilon}_1 I_{n+1}(M\tau) + \frac{1}{(n-2)}\tilde{\varepsilon}_2 I_{n-1}(M\tau) \right. \\ &\quad \left. -\frac{1}{(n+2)}\eta_1 K_{n+1}(M\tau) + \frac{1}{(n-2)}\eta_2 K_{n-1}(M\tau) \right].\end{aligned}\tag{120}$$

We have also to find the basis functions for ghosts. The eigenvalue equations for ghosts in the de Donder gauge have the form

$$\square \varphi_\mu^{(\lambda)} + \lambda \varphi_\mu^{(\lambda)} = 0,$$

and the corresponding fields can be expanded on a family of 3-spheres as

$$\varphi_0(x, \tau) = \sum_{n=1}^{\infty} l_n(\tau) Q^{(n)}(x),\tag{121}$$

$$\varphi_i(x, \tau) = \sum_{n=2}^{\infty} \left[ m_n(\tau) \frac{Q_{|i}^{(n)}(x)}{(n^2 - 1)} + p_n(\tau) S_i^{(n)}(x) \right].\tag{122}$$

The functions  $l_n(\tau)$ ,  $m_n(\tau)$  and  $p_n(\tau)$  can be found similarly to those for harmonics of gravitational perturbations. They have the form

$$l_n(\tau) = \frac{1}{\tau} \left[ \kappa_1 I_{n+1}(M\tau) + \kappa_2 I_{n-1}(M\tau) + \theta_1 K_{n+1}(M\tau) + \theta_2 K_{n-1}(M\tau) \right], \quad (123)$$

$$\begin{aligned} m_n(\tau) &= -(n-1)\kappa_1 I_{n+1}(M\tau) + (n+1)\kappa_2 I_{n-1}(M\tau) \\ &\quad - (n-1)\theta_1 K_{n+1}(M\tau) + (n+1)\theta_2 K_{n-1}(M\tau), \end{aligned} \quad (124)$$

$$p_n(\tau) = \vartheta I_n(M\tau) + \rho K_n(M\tau). \quad (125)$$

## 7 Barvinsky boundary conditions

As we know from section 5, one can set to zero at the boundary the gauge-averaging functional, the whole ghost 1-form, and the perturbation of the induced 3-metric. With the notation of section 5, after making an infinitesimal *gauge* transformation of the metric perturbation  $h_{\mu\nu}$  according to the law (56), one finds in the de Donder gauge (cf. (63))

$$\Phi_\mu^{dD}(h) - \Phi_\mu^{dD}(\widehat{h}) = -\frac{1}{2} \left( \delta_\mu{}^\nu \square + R_\mu{}^\nu \right) \varphi_\nu, \quad (126)$$

where  $R_{\mu\nu}$  is the Ricci tensor of the background, and the elliptic operator on the right-hand side of (126) acts linearly on the ghost 1-form. In our flat Euclidean background, the Ricci tensor vanishes, and on making a 3+1 split of the de Donder functional  $\Phi_\nu^{dD}$  and of the ghost 1-form  $\varphi_\mu$ , the boundary conditions proposed in [36] read as (unlike section 5, we here consider only the de Donder gauge-averaging functional, with the corresponding superscript  $dD$ )

$$\left[ h_{ij} \right]_{\partial M} = \left[ \widehat{h}_{ij} \right]_{\partial M} = 0, \quad (127)$$

$$\left[ \Phi_0^{dD}(h) \right]_{\partial M} = \left[ \Phi_0^{dD}(\widehat{h}) \right]_{\partial M} = 0, \quad (128)$$

$$\left[ \Phi_i^{dD}(h) \right]_{\partial M} = \left[ \Phi_i^{dD}(\widehat{h}) \right]_{\partial M} = 0, \quad (129)$$

$$\left[ \varphi_0 \right]_{\partial M} = 0, \quad (130)$$

$$\left[ \varphi_i \right]_{\partial M} = 0. \quad (131)$$

Once again, the vanishing of the whole ghost 1-form at the boundary ensures the invariance of the boundary conditions (127) under the transformations (56). At that stage, the only remaining set of boundary conditions on metric perturbations, whose invariance under (56) is again guaranteed by (130), (131), is given by (128), (129) by virtue of (126). In this respect, these boundary conditions are the natural generalization of magnetic boundary conditions for Euclidean Maxwell theory, where one sets to zero at the boundary the tangential components of the potential, the gauge-averaging functional, and hence the ghost 0-form. The boundary conditions (127)–(131) were considered in [36] as part of the effort to understand the relation between the wave function of the universe and the effective action in quantum field theory. The loop expansion in quantum cosmology was then obtained after a thorough study of boundary conditions for the propagator.

In light of (127), the boundary conditions (128), (129) lead to mixed boundary conditions on the metric perturbations which take the form (cf. [38, 39, 40])

$$\left[ \frac{\partial h_{00}}{\partial \tau} + \frac{6}{\tau} h_{00} - \frac{\partial}{\partial \tau} (g^{ij} h_{ij}) + \frac{2}{\tau^2} h_{0i}{}^{;i} \right]_{\partial M} = 0, \quad (132)$$

$$\left[ \frac{\partial h_{0i}}{\partial \tau} + \frac{3}{\tau} h_{0i} - \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \right]_{\partial M} = 0. \quad (133)$$

To evaluate the scaling behaviour of the corresponding 1-loop amplitudes, it is necessary to write down the mode-by-mode form of the boundary conditions (132), (133), (127), (130) and (131). They lead to

$$\frac{da_n}{d\tau} + \frac{6}{\tau} a_n - \frac{1}{\tau^2} \frac{de_n}{d\tau} - \frac{2}{\tau^2} b_n = 0 \quad \text{at} \quad \partial M, \quad (134)$$

$$\frac{db_n}{d\tau} + \frac{3}{\tau} b_n - \frac{(n^2 - 1)}{2} a_n = 0 \quad \text{at} \quad \partial M, \quad (135)$$

$$\frac{dc_n}{d\tau} + \frac{3}{\tau} c_n = 0 \quad \text{at} \quad \partial M, \quad (136)$$

$$d_n = 0 \quad \text{at} \quad \partial M, \quad (137)$$

$$e_n = 0 \quad \text{at} \quad \partial M, \quad (138)$$

$$f_n = 0 \quad \text{at} \quad \partial M, \quad (139)$$

$$k_n = 0 \quad \text{at} \quad \partial M, \quad (140)$$

$$l_n = 0 \quad \text{at} \quad \partial M, \quad (141)$$

$$m_n = 0 \quad \text{at} \quad \partial M, \quad (142)$$

$$p_n = 0 \quad \text{at} \quad \partial M. \quad (143)$$

On using, for example, the technique of section 2, the corresponding contributions to  $\zeta(0)$  are found to be (the results quoted below are independent of the particular algorithm used)

$$\zeta(0)_{\text{transverse-traceless modes}} = -\frac{278}{45}, \quad (144)$$

$$\zeta(0)_{\text{partially decoupled modes}} = -2 - 15 = -17, \quad (145)$$

$$\zeta(0)_{\text{vector modes}} = 12 - \frac{11}{60} - \frac{2}{3} - \frac{31}{180} = \frac{494}{45}, \quad (146)$$

$$\zeta(0)_{\text{decoupled vector mode}} = -\frac{15}{2}, \quad (147)$$

$$\zeta(0)_{\text{scalar ghost modes}} = -2 \left( \frac{179}{120} + \frac{59}{360} \right) = -\frac{149}{45}, \quad (148)$$

$$\zeta(0)_{\text{vector ghost modes}} = -2 \left( -\frac{41}{120} - \frac{31}{360} \right) = \frac{77}{90}, \quad (149)$$

$$\zeta(0)_{\text{decoupled ghost mode}} = \frac{5}{2}. \quad (150)$$

## 7.1 Eigenvalue condition for scalar modes

For scalar modes, which are not discussed in the previous list of results, one finds eventually the eigenvalues  $E = X^2$  from the roots  $X$  of [18, 19]

$$J'_n(x) \pm \frac{n}{x} J_n(x) = 0, \quad (151)$$

$$J'_n(x) + \left( -\frac{x}{2} \pm \frac{n}{x} \right) J_n(x) = 0, \quad (152)$$

where  $J_n$  are the Bessel functions of first kind. Note that both  $x$  and  $-x$  solve the same equation.

## 7.2 Four spectral zeta-functions for scalar modes

As we know from section 2, by virtue of the Cauchy theorem and of suitable rotations of integration contours in the complex plane [8, 21], the eigenvalue conditions (151) and (152) give rise to the following four spectral zeta-functions [18, 19],

$$\zeta_{A,B}^{\pm}(s) \equiv \frac{\sin(\pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \int_0^{\infty} dz \frac{\frac{\partial}{\partial z} \log F_{A,B}^{\pm}(zn)}{z^{2s}}, \quad (153)$$

where, denoting by  $I_n$  the modified Bessel functions of the first kind (here  $\beta_+ \equiv n, \beta_- \equiv n+2$ ),

$$F_A^{\pm}(zn) \equiv z^{-\beta_{\pm}} \left( zn I_n'(zn) \pm n I_n(zn) \right), \quad (154)$$

$$F_B^{\pm}(zn) \equiv z^{-\beta_{\pm}} \left( zn I_n'(zn) + \left( \frac{z^2 n^2}{2} \pm n \right) I_n(zn) \right). \quad (155)$$

Regularity at the origin is easily proved in the elliptic sectors, corresponding to  $\zeta_A^{\pm}(s)$  and  $\zeta_B^{\pm}(s)$  [18, 19].

## 7.3 Regularity at the origin of $\zeta_B^+$

With the notation in Refs. [18, 19], if one defines the variable  $\tau \equiv (1+z^2)^{-\frac{1}{2}}$ , one can write the uniform asymptotic expansion of  $F_B^+$  in the form [18, 19]

$$F_B^+ \sim \frac{e^{n\eta(\tau)}}{h(n)\sqrt{\tau}} \frac{(1-\tau^2)}{\tau} \left( 1 + \sum_{j=1}^{\infty} \frac{r_{j,+}(\tau)}{n^j} \right). \quad (156)$$

On splitting the integral  $\int_0^1 d\tau = \int_0^{\mu} d\tau + \int_{\mu}^1 d\tau$  with  $\mu$  small, one gets an asymptotic expansion of the left-hand side of Eq. (153) by writing, in the first interval on the right-hand side,

$$\log \left( 1 + \sum_{j=1}^{\infty} \frac{r_{j,+}(\tau)}{n^j} \right) \sim \sum_{j=1}^{\infty} \frac{R_{j,+}(\tau)}{n^j}, \quad (157)$$

and then computing [18, 19]

$$C_j(\tau) \equiv \frac{\partial R_{j,+}}{\partial \tau} = (1-\tau)^{-j-1} \sum_{a=j-1}^{4j} K_a^{(j)} \tau^a. \quad (158)$$

Remarkably, by virtue of the identity obeyed by the spectral coefficients  $K_a^{(j)}$  on the 4-ball, i.e.

$$g(j) \equiv \sum_{a=j}^{4j} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} K_a^{(j)} = 0, \quad (159)$$

which holds  $\forall j = 1, \dots, \infty$ , one finds [18, 19]

$$\lim_{s \rightarrow 0} s \zeta_B^+(s) = \frac{1}{6} \sum_{a=3}^{12} a(a-1)(a-2) K_a^{(3)} = 0, \quad (160)$$

and [18, 19]

$$\zeta_B^+(0) = \frac{5}{4} + \frac{1079}{240} - \frac{1}{2} \sum_{a=2}^{12} \omega(a) K_a^{(3)} + \sum_{j=1}^{\infty} f(j) g(j) = \frac{296}{45}, \quad (161)$$

where, on denoting here by  $\psi$  the logarithmic derivative of the  $\Gamma$ -function [18, 19],

$$\begin{aligned} \omega(a) \equiv & \frac{1}{6} \frac{\Gamma(a+1)}{\Gamma(a-2)} \left[ -\log(2) - \frac{(6a^2 - 9a + 1)}{4} \frac{\Gamma(a-2)}{\Gamma(a+1)} \right. \\ & \left. + 2\psi(a+1) - \psi(a-2) - \psi(4) \right], \end{aligned} \quad (162)$$

$$f(j) \equiv \frac{(-1)^j}{j!} \left[ -1 - 2^{2-j} + \zeta_R(j-2)(1 - \delta_{j,3}) + \gamma \delta_{j,3} \right]. \quad (163)$$

Equation (159) achieves three goals:

- (i) Vanishing of the  $\log(2)$  coefficient in (161);
- (ii) Vanishing of  $\sum_{j=1}^{\infty} f(j) g(j)$  in (161);
- (iii) Regularity at the origin of  $\zeta_B^+$ .

## 7.4 Interpretation of the result

Since all other  $\zeta(0)$  values for pure gravity obtained in the literature on the 4-ball are negative, the analysis here briefly outlined shows that only fully diffeomorphism-invariant boundary conditions lead to a positive  $\zeta(0)$

value for pure gravity on the 4-ball, and hence *only fully diffeomorphism-invariant boundary conditions lead to a vanishing cosmological wave function for vanishing 3-geometries at 1-loop level, at least on the Euclidean 4-ball*. If the probabilistic interpretation is tenable for the whole universe, this means that *the universe has vanishing probability of reaching the initial singularity at  $a = 0$* , which is therefore avoided by virtue of quantum effects [18, 19], since the 1-loop wave function is proportional to  $a^{\zeta(0)}$  [7].

## 8 The strong ellipticity issue

The result outlined in section 7 is non-trivial because the zeta-function  $\zeta_B^+$  corresponds to the sector of the boundary-value problem for which strong ellipticity [26] fails to hold [39, 40]. We now define in detail this concept, and we are also going to discuss its relevance both for physics and mathematics.

Let  $M$  be a smooth, compact Riemannian manifold endowed with a positive-definite metric  $g$  and assume that the boundary  $\partial M$  is smooth. Let  $V$  be a vector bundle over  $M$  and  $C^\infty(V, M)$  be the space of smooth sections of the bundle  $V$ . With the introduction of a Hermitian metric  $E$  and the Riemannian volume element on  $M$ , the dual bundle  $V^*$  can be identified with  $V$  and a natural inner product for the smooth sections of  $V$  can be defined. It is clear that the Hilbert space  $\mathcal{L}^2(V, M)$  is identified with the completion of  $C^\infty(V, M)$  with respect to the inner product. An operator of Laplace type is a map

$$\mathcal{L} : C^\infty(V, M) \longrightarrow C^\infty(V, M) , \quad (164)$$

expressed as follows:

$$\mathcal{L} = -g^{ab}\nabla_a^V\nabla_b^V + Q , \quad (165)$$

where  $\nabla^V$  denotes the connection on  $V$  and  $Q$  represents a self-adjoint endomorphism of  $V$ .

The boundary data for the Laplace operator under consideration can be written as

$$\psi_{\mathcal{L}}(\varphi) = \begin{pmatrix} \psi_0(\varphi) \\ \psi_1(\varphi) \end{pmatrix} , \quad (166)$$

where  $\varphi \in C^\infty(V, M)$  and we have set

$$\psi_0(\varphi) = \varphi|_{\partial M} , \quad \psi_1(\varphi) = \nabla_N \varphi|_{\partial M} , \quad (167)$$



with  $\nabla_N$  denoting the normal covariant derivative with respect to the boundary  $\partial M$ . By introducing the tangential differential operator  $B_{\mathcal{L}}$  on  $\partial M$  the boundary conditions can be written in a concise way as

$$B_{\mathcal{L}}\psi_{\mathcal{L}}(\varphi) = 0 , \quad (168)$$

where the general form of  $B_{\mathcal{L}}$ , which ensures the self-adjointness of the operator  $\mathcal{L}$ , is [38, 39, 40]

$$B_{\mathcal{L}} = \begin{pmatrix} \Pi & 0 \\ \Lambda & (\mathbb{I} - \Pi) \end{pmatrix} , \quad (169)$$

where  $\Pi$  denotes a self-adjoint projector and  $\Lambda$  is a self-adjoint tangential differential operator satisfying the relation

$$\Pi\Lambda = \Lambda\Pi = 0 , \quad (170)$$

and  $\mathbb{I}$  is the identity endomorphism of  $V$ . The operator  $\Lambda$  can always be cast in the manifestly self-adjoint form

$$\Lambda = (\mathbb{I} - \Pi) \left\{ \frac{1}{2} \left( \Gamma^i \hat{\nabla}_i + \hat{\nabla}_i \Gamma^i \right) + S \right\} (\mathbb{I} - \Pi) , \quad (171)$$

where  $\hat{\nabla}_i$  represents the covariant tangential derivative, compatible with the induced metric on  $\partial M$ , and  $\Gamma^i$  and  $S$  are endomorphisms satisfying the relations

$$\bar{\Gamma}^i = E^{-1}(\Gamma^i)^\dagger E = -\Gamma^i , \quad \bar{S} = E^{-1}S^\dagger E = S , \quad (172)$$

$$\Pi\Gamma^i = \Gamma^i\Pi = \Pi S = S\Pi = 0 . \quad (173)$$

Here, the bar denotes the adjoint in the space  $\mathcal{L}^2(V, M)$  and dagger is the Hermitian conjugate.

It is instructive to notice that different choices for the projector  $\Pi$  and the operator  $\Lambda$  lead to different types of boundary conditions. More precisely: for  $\Pi = \mathbb{I}$  and  $\Lambda = 0$  one obtains Dirichlet boundary conditions, for  $\Pi = \Lambda = 0$  one recovers Neumann boundary conditions, and for  $\Pi = 0$  and  $\Lambda \neq 0$ , with  $\Lambda$  not a differential operator, the boundary conditions are reduced to the Robin type.

The leading symbol of the Laplace operator (165) is defined as

$$\sigma_L(\mathcal{L}; x, \xi) \equiv |\xi|^2 \cdot \mathbb{I} = g^{ab}(x) \xi_a \xi_b \cdot \mathbb{I} , \quad (174)$$

where  $\xi \in T^*M$  is an arbitrary cotangent vector. For a non-singular Riemannian metric the leading symbol (171) is positive-definite and, hence, the operator  $\mathcal{L}$  is elliptic [41]. In order to analyze the strong ellipticity condition for the boundary-value problem (165) and (168) we need to introduce the leading symbol of the boundary-value operator  $B_{\mathcal{L}}$ . Such leading symbol, denoted by  $\sigma_g(B_{\mathcal{L}})$ , is defined as [40]

$$\sigma_g(B_{\mathcal{L}}) = \begin{pmatrix} \Pi & 0 \\ iT & (\mathbb{I} - \Pi) \end{pmatrix}, \quad (175)$$

where, by exploiting the relation (171), one has

$$T = -i\sigma_L(\Lambda) = \Gamma^i \rho_i, \quad (176)$$

with an arbitrary cotangent vector  $\rho$  on  $T^*(\partial M)$ , the cotangent bundle of the boundary of  $M$ . From the relations (172) one can prove that the matrix  $T$  defined above is anti-self-adjoint,

$$\bar{T} = -T, \quad (177)$$

and it satisfies the equation

$$\Pi T = T \Pi = 0. \quad (178)$$

In a neighbourhood of  $\partial M$  the Riemannian manifold  $M$  can be locally described by a direct product  $\Omega = [0, \epsilon] \times \partial M$ . A local set of coordinates for  $\Omega$  is  $x^\mu = x^\mu(r, \hat{x}^i)$  where  $r$  denotes the normal distance from the boundary and  $\hat{x}^i$  are the coordinates on the *moved* boundary  $\partial M(r) = \{x \in M \mid r(x) = r\}$  with  $r \in [0, \epsilon]$ . In this setting, the leading symbol of the Laplace operator  $\mathcal{L}$  can be written as  $\sigma_L(\mathcal{L}; \hat{x}, r, \rho, \omega)$ . Let us set  $r = 0$ , make the replacement  $\omega \rightarrow -i\partial_r$ , and consider the resulting differential equation

$$[\sigma_L(\mathcal{L}; \hat{x}, 0, \rho, -i\partial_r) - \lambda \cdot \mathbb{I}] \phi(r) = 0, \quad (179)$$

with  $\lambda \in \mathbb{C} - \mathbb{R}^+$ , and whose solutions must satisfy the asymptotic condition

$$\lim_{r \rightarrow \infty} \phi(r) = 0. \quad (180)$$

The boundary-value problem, consisting of the pair (165) and (168), is said to be *strongly elliptic* with respect to the cone  $\mathbb{C} - \mathbb{R}^+$  if, for any  $\rho \in$

$T^*(\partial M)$ ,  $\lambda \in \mathbb{C} - \mathbb{R}^+$  and  $\psi'_{\mathcal{L}}$ , there exists a unique solution to the equation (179) satisfying both the asymptotic condition (180) and the relation

$$\sigma_g(B_{\mathcal{L}})(\hat{x}, \rho)\psi_{\mathcal{L}}(\phi) = \psi'_{\mathcal{L}}(\phi) . \quad (181)$$

For an operator of Laplace type the differential equation (179) can be written as

$$[-\partial_r^2 + |\rho|^2 - \lambda] \phi(r) = 0 . \quad (182)$$

The general solution to the above equation which satisfies the condition (180) of decay as  $r \rightarrow \infty$  is

$$\phi(r) = \chi \exp(-\mu r) , \quad (183)$$

where we have set  $\mu = \sqrt{|\rho|^2 - \lambda}$ . The boundary data for the solution (183) can be expressed as

$$\psi_{\mathcal{L}}(\phi) = \begin{pmatrix} \chi \\ -\mu\chi \end{pmatrix} . \quad (184)$$

The boundary-value problem under consideration is strongly elliptic if the boundary data (184) satisfy equation (181). More precisely, strong ellipticity holds if the matrix associated with the linear system

$$\begin{pmatrix} \Pi & 0 \\ iT & (\mathbb{I} - T) \end{pmatrix} \begin{pmatrix} \chi \\ -\mu\chi \end{pmatrix} = \begin{pmatrix} \psi'_0 \\ \psi'_1 \end{pmatrix} \quad (185)$$

is invertible. One can show that the above system is equivalent to the set

$$\begin{aligned} \Pi\chi &= \psi'_0, \\ (\mu\mathbb{I} - iT)\chi &= \mu\psi'_0 - \psi'_1 . \end{aligned} \quad (186)$$

Since the first equation in (186) is independent of the second [40], verifying the invertibility of the matrix in (185), and hence strong ellipticity, is equivalent to verifying the existence of a unique solution to the second equation in (186) for arbitrary  $\psi'_0$  and  $\psi'_1$ . A necessary and sufficient condition for the existence of a unique solution to the second equation in (186) can be found to be

$$\det [\mu\mathbb{I} - iT] \neq 0 . \quad (187)$$

The matrix  $iT$  defined in (176) is self-adjoint, and therefore its eigenvalues are real. It is clear that for any  $\lambda \in \mathbb{C} - \mathbb{R}_+$  the quantity  $\mu = \sqrt{|\rho|^2 - \lambda}$  is complex and, hence, the matrix  $[\mu\mathbb{I} - iT]$  is non-degenerate. For  $\lambda \in \mathbb{R}_-$  the

quantity  $\mu$  is real and satisfies the inequality  $\mu > |\rho|$ . The above remarks together with the condition (187) imply that

$$|\rho|\mathbb{I} - iT > 0 . \quad (188)$$

By noticing that  $(|\rho|\mathbb{I} - iT)(|\rho|\mathbb{I} + iT) = \mathbb{I}|\rho|^2 + T^2$  we can conclude that the boundary-value problem (165) and (168) is *strongly elliptic* if the eigenvalues of the matrix  $T^2$  are real and greater than  $-|\rho|^2$ , i.e.

$$\Im(T^2) = 0 , \quad \Re(\mathbb{I}|\rho|^2 + T^2) > 0 , \quad (189)$$

for any  $\rho \in T^*(\partial M)$ .

In the setting of Euclidean quantum gravity it has been shown [39, 40] that the eigenvalues of the matrix  $T$  are the following:

$$\text{spec}(T) = \begin{cases} 0 & \text{with degeneracy } [m(m+1)/2 - 2] \\ i\rho & \text{with degeneracy } 1 \\ -i\rho & \text{with degeneracy } 1 . \end{cases} \quad (190)$$

Since the eigenvalues of the matrix  $T^2$  are 0 and  $-|\rho|^2$  the strong ellipticity condition (189) is *not* satisfied. This means, in particular, that the dynamical operator of the metric perturbations endowed with diffeomorphism-invariant boundary conditions in the de Donder gauge is not strongly elliptic.

## 9 Concluding remarks and open problems

In order to avoid the problem of the lack of strong ellipticity in Euclidean quantum gravity one can consider various alternative approaches.

When deriving the operator that describes the dynamics of the metric perturbations the particular choice of the de Donder gauge renders the operator of Laplace type but leads to the lack of strong ellipticity of the associated boundary-value problem. It remains to be seen whether strong ellipticity can be preserved if one considers instead dynamical operators on metric perturbations which are non-minimal.

Another approach is to study Euclidean quantum gravity with non-local boundary conditions [42], or with boundary conditions which are not gauge invariant and that eliminate the occurrence of tangential derivatives [43].

Although viable, the alternative approaches mentioned above contain some problems and difficulties. It is, therefore, still unclear what is the

most appropriate way to solve the problem of the lack of strong ellipticity in Euclidean quantum gravity. The result of section 7, however, shows that there exists at least one background where a meaningful  $\zeta(0)$  value is still obtainable despite the lack of strong ellipticity. At a deeper mathematical level, strong ellipticity makes it possible to define the heat operator, which is however not necessary in order to define the resolvent or complex powers of the given elliptic operator. For the latter two, one needs just a sector of the complex plane free of eigenvalues of the leading symbol. [We are grateful to Gerd Grubb for correspondence about this issue]. The investigation of other backgrounds might provide further examples of meaningful  $\zeta(0)$  values despite violation of strong ellipticity, and their relevance for quantum cosmology and/or quantum field theory should be assessed.

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